

Lecture 2: Girth, Connectivity and Bipartite Graphs

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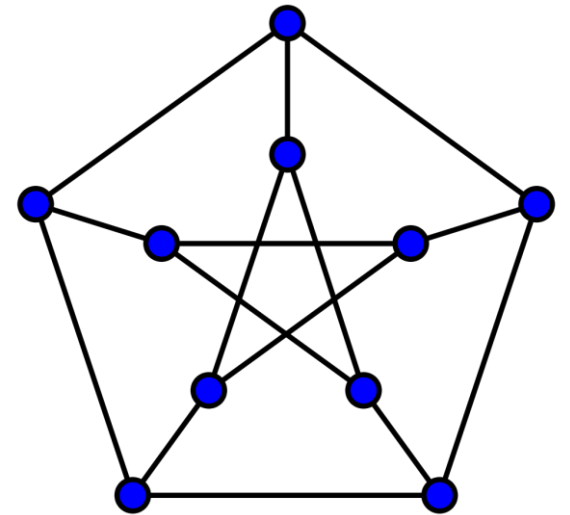
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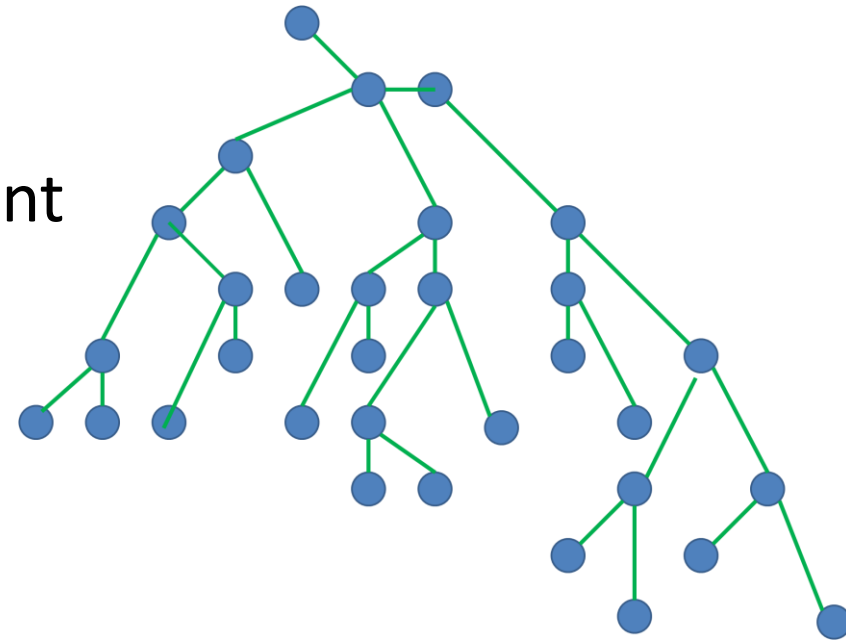
Girth

- The minimum length of a cycle in a graph G is the **girth** $g(G)$ of G
- Example: The Peterson graph is the unique **5-cage**
 - cubic graph (every vertex has degree 3)
 - girth = **5**
 - smallest graph satisfies the above properties



Girth (cont.)

- A tree has girth ∞
- Note that a tree can be colored with two different colors
- \Rightarrow A graph with large girth has small chromatic number?
- Unfortunately NO!
- Theorem (Erdős, 1959) For all k, l , there exists a graph G with $g(G) > l$ and $\chi(G) > k$



Girth and diameter

- **Proposition** (1.3.2, D) Every graph G containing a cycle satisfies $g(G) \leq 2 \operatorname{diam}(G) + 1$
- When the equality holds?

Girth and minimal degree lower bounds graph size

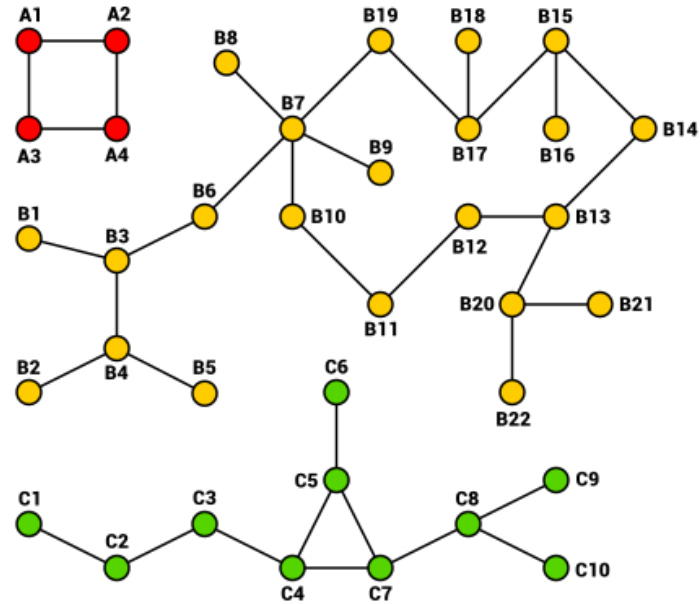
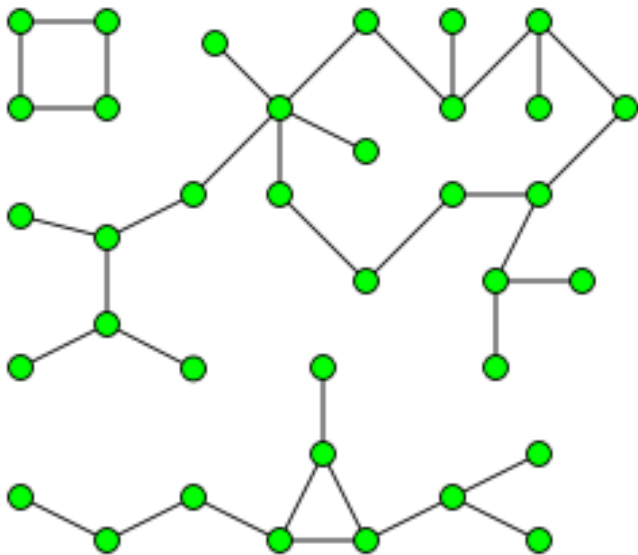
- $n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r \text{ is even} \end{cases}$
- **Exercise** (Ex7, ch1, D) Let G be a graph. If $\delta(G) \geq \delta \geq 2$ and $g(G) \geq g$, then $|G| \geq n_0(\delta, g)$
- **Corollary** (1.3.5, D) If $\delta(G) \geq 3$, then $g(G) < 2 \log_2 |G|$

Triangle-free upper bounds # of edges

- **Theorem** (1.3.23, W, Mantel 1907) The maximum number of edges in an n -vertex triangle-free simple graph is $\lfloor n^2/4 \rfloor$
- The bound is best possible
- There is a triangle-free graph with $\lfloor n^2/4 \rfloor$ edges: $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$
- Extremal problems

Connected, connected component

- A graph G is **connected** if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of G is a **(connected) component**



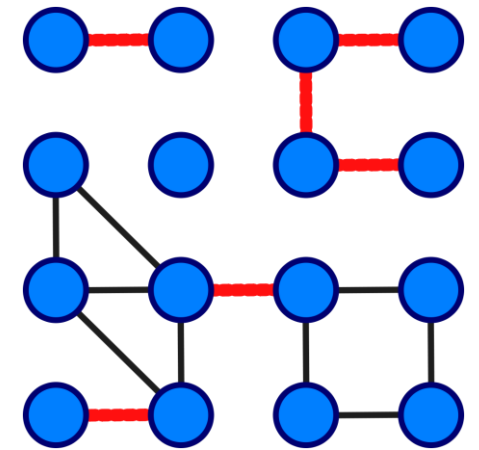
Quiz

- **Problem** (1B, L) Suppose G is a graph on 10 vertices that is not connected. Prove that G has at most 36 edges. Can equality occur?
- **More general** (Ex9, S1.1.2, H) Let G be a graph of order n that is not connected. What is the maximum size of G ?

Connected vs. minimal degree

- **Proposition** (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then G is connected
- (Ex16, S1.1.2, H; 1.3.16, W)
If $\delta(G) \geq \frac{n-2}{2}$, then G need not be connected
- Extremal problems
- “best possible” “sharp”

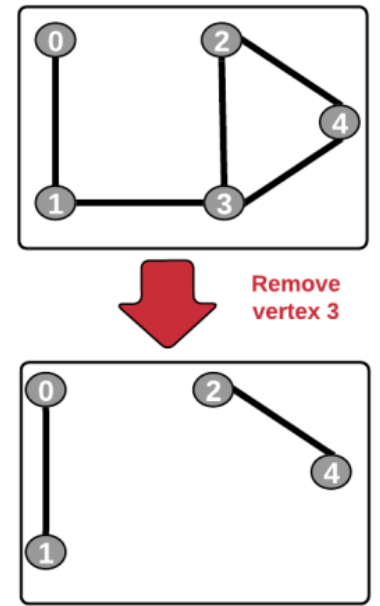
Add/delete an edge



- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
 - \Rightarrow deleting an edge increases the number of components by 0 or 1
- **Proposition** (1.2.11, W)
Every graph with n vertices and k edges has at least $n - k$ components
- An edge e is called a **bridge** if the graph $G - e$ has more components
- **Proposition** (1.2.14, W)
An edge e is a bridge $\iff e$ lies on no cycle of G
 - Or equivalently, an edge e is not a bridge $\iff e$ lies on a cycle of G

Cut vertex and connectivity

- A node v is a **cut vertex** if the graph $G - v$ has more components
- A proper subset S of vertices is a **vertex cut set** if the graph $G - S$ is disconnected, or trivial (a graph of order 0 or 1)
- The **connectivity**, $\kappa(G)$, is the minimum size of a cut set of G
 - The graph is k -connected for any $k \leq \kappa(G)$



Connectivity properties

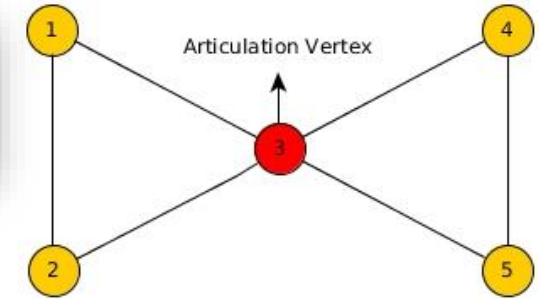
- $\kappa(K^n) = n - 1$
- If G is disconnected, $\kappa(G) = 0$
 - \Rightarrow A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n , then
$$1 \leq \kappa(G) \leq n - 2$$

Connectivity properties (cont.)

Proposition (1.2.14, W)

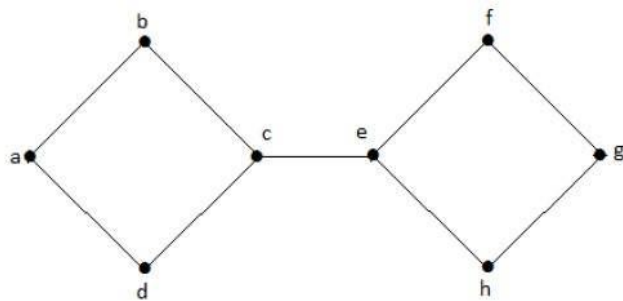
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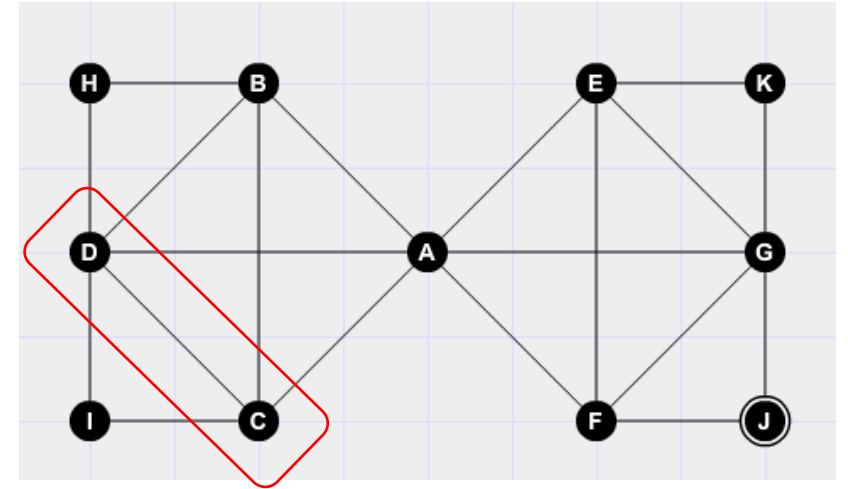
- $\kappa(G) \geq 2 \Leftrightarrow G$ is connected and has no cut vertices
- A vertex lies on a cycle \nRightarrow it is not a cut vertex
 - \Rightarrow (Ex13, S1.1.2, H) Every vertex of a connected graph G lies on at least one cycle $\nRightarrow \kappa(G) \geq 2$
 - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

- (Ex12, S1.1.2, H) G has a cut vertex vs. G has a bridge



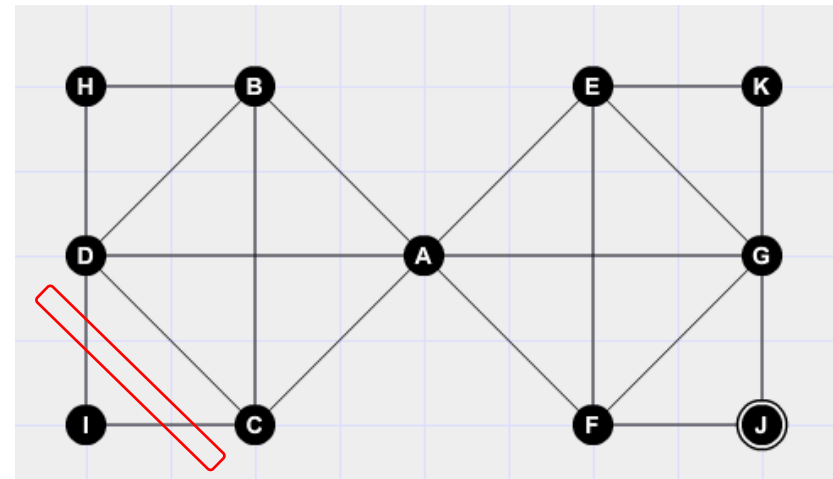
Connectivity and minimal degree

- (Ex15, S1.1.2, H)
- $\kappa(G) \leq \delta(G)$
- If $\delta(G) \geq n - 2$, then $\kappa(G) = \delta(G)$



Edge-connectivity

- A proper subset $F \subset E$ is edge cut set if the graph $G - F$ is disconnected
- The **edge-connectivity** $\lambda(G)$ is the minimal size of edge cut set
- $\lambda(G) = 0$ if G is disconnected
- **Proposition** (1.4.2, D) If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$

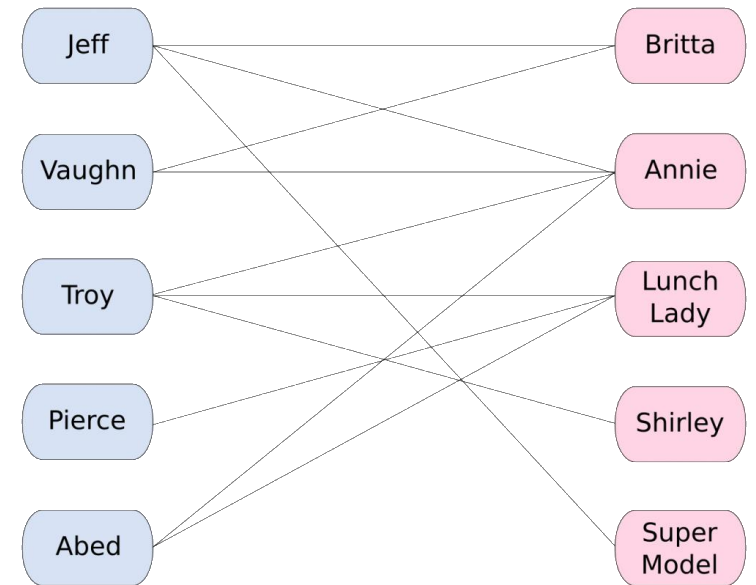


Large average (minimal) degree implies local large connectivity

- **Theorem** (1.4.3, D, Mader 1972) Every graph G with $d(G) \geq 4k$ has a $(k + 1)$ -connected subgraph H such that $d(H) > d(G) - 2k$.

Bipartite graphs

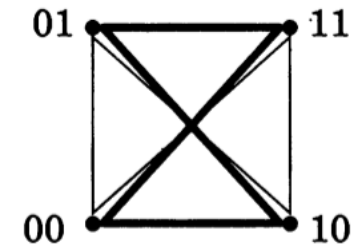
- **Theorem** (1.2.18, W, König 1936)
A graph is bipartite \iff it contains no odd cycle



Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Complete graph is a union of bipartite graphs

- The **union** of graphs G_1, \dots, G_k , written $G_1 \cup \dots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^k V(G_i)$ and edge set $\bigcup_{i=1}^k E(G_i)$
- Consider an air traffic system with k airlines
 - Each pair of cities has direct service from at least one airline
 - No airline can schedule a cycle through an odd number of cities
 - Then, what is the maximum number of cities in the system?
- **Theorem** (1.2.23, W) The complete graph K_n can be expressed as the union of k bipartite graphs $\Leftrightarrow n \leq 2^k$



Bipartite subgraph is large

- **Theorem** (1.3.19, W) Every loopless graph G has a bipartite subgraph with at least $|E|/2$ edges

Summary

- Girth
 - Girth vs diameter
 - Girth and minimal degree lower bounds graph size
 - Girth > 3 upper bounds # of edges
- Connectivity
 - Connected components
 - Bridge/cut vertex/connectivity/edge-connectivity
 - Minimal degree and connectivity
 - $\kappa(G) \leq \lambda(G) \leq \delta(G)$
 - Large average (minimal) degree implies local large connectivity
- Bipartite graphs
 - Equivalent to containing no odd cycle
 - Every graph can be decomposed as a union of bipartite graphs, with one large enough

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Questions?